

Extra Practice Problems 2

Here's a set of a bunch of practice problems you can work through to help prepare for the second midterm exam. These topics span all of the discrete math topics we've covered so far, so feel free to focus on whichever problems seem most interesting.

Problem One: Slicing an Orange

You have a perfectly spherical orange with five stickers on it. Prove that there is some way to slice the orange into two equal halves so that one of the halves has pieces of at least four of the stickers on it.

Problem Two: Inductive Sets

A set S is called an *inductive set* if the follow two properties are true about S :

- $0 \in S$.
- For any number $x \in S$, the number $x + 1$ is also an element of S .

This question asks you to explore various properties of inductive sets.

- Find two different examples of inductive sets.
- Prove that the intersection of any two inductive sets is also an inductive set.
- Prove that if S is an inductive set, then $\mathbb{N} \subseteq S$.
- Prove that \mathbb{N} is the *only* inductive set that's a subset of all inductive sets. This proves that \mathbb{N} is, in a sense, the most “fundamental” inductive set. In fact, in foundational mathematics, the set \mathbb{N} is sometimes defined as the one inductive set that's a subset of all inductive sets.

Problem Three: Odd and Even Functions

Up to this point, most of our discussion of functions has involved functions from arbitrary domains to arbitrary codomains. If we restrict ourselves to functions with specific types of domains and codomains, then we can start exploring more nuanced and interesting classes of functions.

Let's suppose that we have a function $f : \mathbb{R} \rightarrow \mathbb{R}$. We'll say that f is an **odd function** if the following is true:

$$\forall x \in \mathbb{R}. f(-x) = -f(x)$$

This function explores properties of odd functions.

- i. Prove that if $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ are odd, then $g \circ f$ is also odd.
- ii. Prove that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is odd and is a bijection, then f^{-1} is also odd.

We can define **even functions** as follows. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called **even** if the following is true:

$$\forall x \in \mathbb{R}. f(-x) = f(x)$$

- iii. Prove that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is an even function, then f is **not** a bijection.

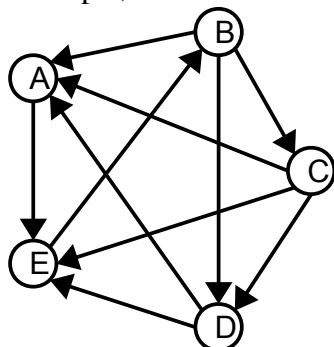
It turns out that every function $f : \mathbb{R} \rightarrow \mathbb{R}$ can be written as the sum of an odd function and an even function. The next few parts of this problem ask you to prove this.

- iv. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an odd function. Prove that for any $r \in \mathbb{R}$, the function $r \cdot f : \mathbb{R} \rightarrow \mathbb{R}$ defined as $(r \cdot f)(x) = r \cdot f(x)$ is also odd.
- v. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an even function. Prove that for any $r \in \mathbb{R}$, the function $r \cdot f : \mathbb{R} \rightarrow \mathbb{R}$ defined as $(r \cdot f)(x) = r \cdot f(x)$ is also even.
- vi. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be any function. Prove that $g : \mathbb{R} \rightarrow \mathbb{R}$ defined as $g(x) = f(x) - f(-x)$ is odd.
- vii. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be any function. Prove that $h : \mathbb{R} \rightarrow \mathbb{R}$ defined as $h(x) = f(x) + f(-x)$ is even.
- viii. Prove that any function $f : \mathbb{R} \rightarrow \mathbb{R}$ can be expressed as the sum of an odd function and an even function.

Problem Four: Tournament Graphs and Binary Relations

Let's quickly refresh a definition. A *tournament* is a contest between some number of players in which each player plays each other player exactly once. We assume that no games end in a tie, so each game ends in a win for one of the players.

Here's a new definition to work with. If p is a player in tournament T , then we can define the set $W(p) = \{x \mid x \text{ is a player in } T \text{ and } p \text{ beat } x\}$. Intuitively, $W(p)$ is the set of all the players that player p beat. For example, in the tournament on the left, $W(B) = \{A, C, D\}$.



Now, let's define a new binary relation. Let T be a tournament. We'll say that $p_1 \sqsubset_T p_2$ if $W(p_1) \subsetneq W(p_2)$. Intuitively, $p_1 \sqsubset_T p_2$ means that p_2 beat every player that p_1 beat, plus some additional players.

For example, in the tournament to the left, we have that $D \sqsubset_T C$ because $W(D) = \{A, E\}$ and $W(C) = \{A, D, E\}$. Similarly, we know $A \sqsubset_T D$ since $W(A) = \{E\}$ and $W(D) = \{A, E\}$.

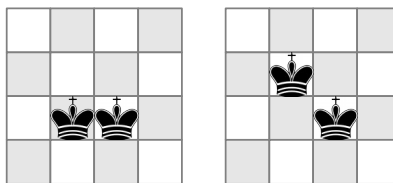
Prove that if T is a tournament, then \sqsubset_T is a strict order over the players in T .

Problem Five: Tournament Graphs and Hamiltonian Paths

A *tournament graph* is a directed graph of n nodes where every pair of distinct nodes has exactly one edge between them. A *Hamiltonian path* is a path in a graph that passes through every node in a graph exactly once. Prove that every tournament graph has a Hamiltonian path. For the purposes of this problem you can consider the *empty path* of no nodes to be a Hamiltonian path through the empty graph.

Problem Six: A Clash of Kings

Chess is a game played on an 8×8 grid with a variety of pieces. In chess, no two king pieces can ever occupy two squares that are immediately adjacent to one another horizontally, vertically, or diagonally. For example, the following positions are illegal:



Prove that it is impossible to legally place 17 kings onto a chessboard.

Problem Seven: Induction and Strict Orders

Let A be a set and $<_A$ be a strict order over A . Recall from Problem Set Four that a *chain in $<_A$* is a series of elements x_1, \dots, x_n drawn from A such that

$$x_1 <_A x_2 <_A \dots <_A x_n.$$

Prove, by induction, that if x_1, \dots, x_n is a chain in $<_A$ with $n \geq 2$ elements, then $x_1 <_A x_n$.

Problem Eight: Strengthening Relations

Let's introduce a new definition. Let R and T be binary relations over the same set A . We'll say that R is *no stronger than* T if the following statement is true:

$$\forall a \in A. \forall b \in A. (aRb \rightarrow aTb)$$

- i. Let R and T be binary relations over the same set A where R is no stronger than T . Prove or disprove: if R is an equivalence relation, then T is an equivalence relation.
- ii. Let R and T be binary relations over the same set A where R is no stronger than T . Prove or disprove: if T is an equivalence relation, then R is an equivalence relation.

Problem Nine: More Fun With Friends and Strangers

(From the Fall 2013 midterm exam)

Suppose you have a 17-clique (that is, an undirected graph with 17 nodes where there's an edge between each pair of nodes) where each edge is colored one of *three* different colors (say, red, green, and blue). Prove that regardless of how the 17-clique is colored, it must contain a blue triangle, a red triangle, or a green triangle. (Hint: Use the theorem on friends and strangers.)

Problem Ten: Bijections and Induction

(From the Fall 2014 midterm exam)

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function. We'll say that f is *linearly bounded* if $f(n) \leq n$ for all $n \in \mathbb{N}$.

Prove that if $f : \mathbb{N} \rightarrow \mathbb{N}$ is linearly bounded and is a bijection, then $f(n) = n$ for all $n \in \mathbb{N}$. (Hint: You might find induction useful here.)

A good question to ponder: is this result still true if we replace the codomain of \mathbb{N} with \mathbb{Z} ? If so, why? If not, why not? And if the answer is no, what specific claim can you point to in your proof that is no longer true?

Problem Eleven: Odd Rational Numbers

Let's say that a rational number r is an **odd rational number** if there exist integers p and q where $r = p/q$ and q is odd. For example, the number 1.6 is an odd rational number because it can be written as $8/5$.

- i. To help you get more familiar with the definition, prove that $3/2$ is not an odd rational number. (*Hint: Read the definition closely. What exactly do you need to prove here?*)

Consider the following binary relation \sim over the set \mathbb{R} :

$$x \sim y \quad \text{if} \quad y - x \text{ is an odd rational number.}$$

- ii. Prove that \sim is an equivalence relation.
- iii. What is $[0]_{\sim}$? Express your answer as simply as possible.
- iv. Prove that every element of $[\sqrt{2}]_{\sim}$ is irrational.

Problem Twelve: Long Paths

(From the Fall 2016 midterm exam)

Let $G = (V, E)$ be a graph where every node has degree at least k for some $k \geq 1$. Let P be a simple path in G that has length less than k . Prove that P is **not** the longest simple path in G .

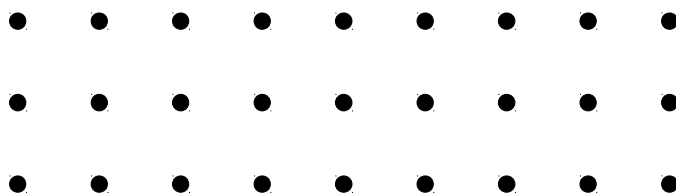
Problem Thirteen: Least and Greatest Elements

Let $<_A$ be a strict order over a set A . We say that an element x is a **least element of $<_A$** if for every element $y \in A$ other than x , the relation $x <_A y$ holds. We say that an element x is a **greatest element of $<_A$** if for every element $y \in A$ other than x , the relation $y <_A x$ holds.

- i. Give an example of a strict order relation with no least or greatest element. Briefly justify your answer.
- ii. Give an example of a strict order relation with a least element but no greatest element. Briefly justify your answer.
- iii. Give an example of a strict order relation with a greatest element but no least element. Briefly justify your answer.
- iv. Give an example of a strict order with a greatest element and a least element. Briefly justify your answer.
- v. Prove that every strict order has at most one greatest element.

Problem Fourteen: Coloring a Grid

You are given a 3×9 grid of points, like the one shown below:



Suppose that you color each point in the grid either red or blue. Prove that no matter how you color those points, you can always find four points of the same color that form the corners of a rectangle.

A good follow-up question: is a 3×9 grid the smallest grid that guarantees a rectangle?

Problem Fifteen: The Six-Color Theorem

In lecture, we talked about the *four-color theorem*, which says that every planar graph is 4-colorable. The proof of the four-color theorem is an incredible exercise in proof by cases, with computers automatically checking each case.

Although the four-color theorem required computers to prove, it's possible to prove a slightly weaker result without such aid: every planar graph is 6-colorable. This is called the *six-color theorem*.

Prove the six-color theorem. You may want to use the following fact, which you don't need to prove: every planar graph with at least one node has a node with degree five or less.

Problem Sixteen: Trees

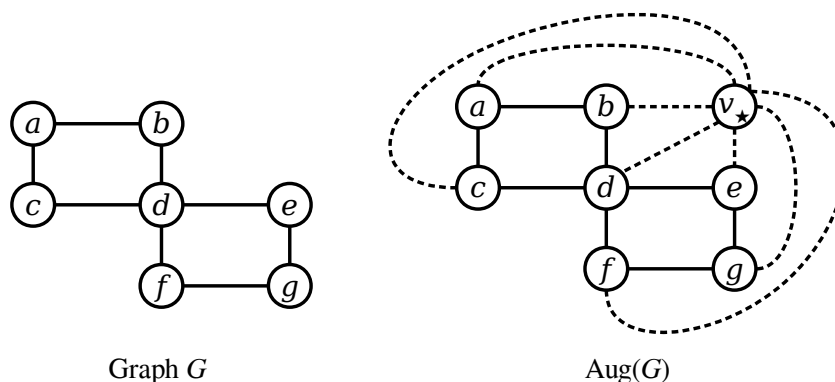
Recall from lecture that a *tree* is an undirected, connected graph with no cycles.

A *leaf* in a tree is a node in a tree whose degree is exactly one.

- i. Prove that any tree with at least two nodes has at least one leaf.
- ii. In lecture, we used complete induction to prove that any tree with $n \geq 1$ nodes has exactly $n-1$ edges. Using your result from part (i), prove this result using only standard induction.

Problem Seventeen: Outerplanar Graphs

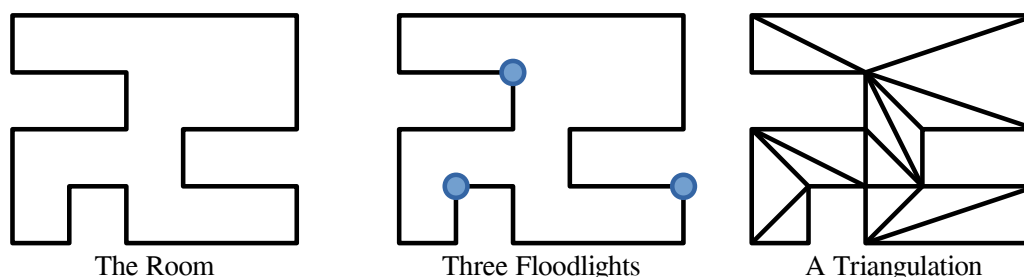
If G is a graph, the *augmentation* of G , denoted $\text{Aug}(G)$, is formed by adding a new node v_\star to G , then adding edges from v_\star to each other node in G . For example, below is a graph G and its augmentation $\text{Aug}(G)$. To make it easier to see the changes between G and $\text{Aug}(G)$, we've drawn the edges added in $\text{Aug}(G)$ using dashed lines:



Here's one more definition: an undirected graph G is called an *outerplanar graph* if $\text{Aug}(G)$ is a planar graph. In other words, if $\text{Aug}(G)$ is a *planar* graph, then the original graph G is an *outerplanar* graph.

- i. Using the four-color theorem about planar graphs, prove the *three-color theorem*: every outerplanar graph is 3-colorable.

Here's a nifty application of outerplanar graphs. Imagine that you have a room in the shape of a polygon. You're interested in placing floodlights in some number of the corners of the room so that the entire room will be illuminated. You can always illuminate the entire room by putting floodlights in all the corners of the room, and the challenge is to find a way to minimize the number of necessary lights. For example, here's one possible room and one set of three floodlights that would illuminate the room:



A useful concept for modeling this problem is *polygon triangulation*. Given a polygon, a triangulation of that polygon is a way of adding extra internal lines connecting the existing vertices of that polygon so that (1) the polygon ends up subdivided into non-overlapping triangles and (2) no new vertices are added. One possible triangulation of the original room is shown above. Importantly, *any floodlight placed at the corner of a triangle will illuminate everything in that triangle*, since there's nothing to obstruct the light.

You can think about the triangulation of a polygon as a planar graph: each vertex is a node, and each line is an edge. But more than that, the triangulation of any polygon is an *outerplanar* graph, since the augmentation is always planar. (You don't need to prove this)

- ii. Using your result from part (i) and the fact that any polygon can be triangulated, prove that you can always illuminate a room in the shape of an n -vertex polygon with at most $\lfloor n/3 \rfloor$ floodlights. (*Hint: If you have a 3-coloring of a triangulated polygon, what must be true about any triangle's corners?*)

Problem Eighteen: Forced Connectivity

Let $G = (V, E)$ be a graph with n nodes. Prove that if the degree of every node in V is at least $(n-1) / 2$, then G is connected.

Problem Nineteen: Lattice Points

A *lattice point* in 2D space is a point whose (x, y) coordinates are integers. For example, $(137, -42)$ is a lattice point, but $(1.5, \pi)$ isn't.

Suppose that you pick any five lattice points in 2D space. Prove that there must be some pair of points in the group with the following property: the midpoint of the line connecting those points is also a lattice point.

Problem Twenty: Planar Graphs

Recall from lecture that a *planar graph* is a graph that can be drawn in the 2D plane such that no two edges cross one another. The *four-color theorem* says that all planar graphs are 4-colorable.

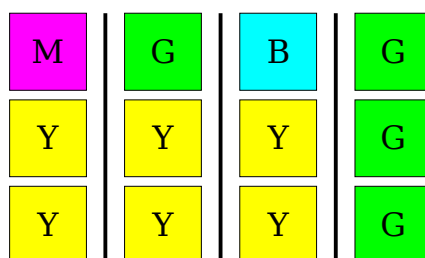
A *k-clique* is a graph consisting of k nodes that are all adjacent to one another. Prove that the 5-clique is not planar.

Problem Twenty-One: Colored Cubes*

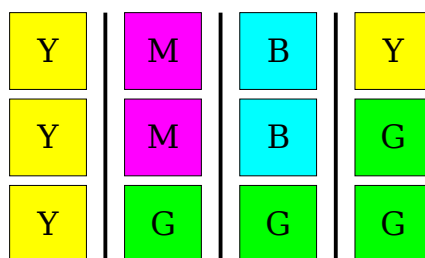
Suppose that you have a collection of cubes of n different colors. For simplicity we'll assume that the total number of cubes you have is a multiple of n ; specifically, let's suppose that you have kn total cubes, where k is some natural number. For example, you might have 30 cubes of six different colors, in which case $n = 6$ and $k = 5$. Alternatively, you might have 200 cubes of 40 different colors, where $n = 40$ and $k = 5$.

Now, let's suppose that you have n bins into which you can place the cubes, each of which holds exactly k different cubes. Although it may not seem like it, it's always possible to distribute the cubes into the boxes such that every box is full (that is, it has exactly k cubes in it) and that each box has cubes of at most two different colors. Prove this fact using induction on n , the number of colors.

Some examples might help here. Suppose that $n = 4$ and $k = 3$, meaning that there are four different colors of cubes, twelve total cubes, and four boxes that hold three cubes each. The goal is then to put the cubes into the four boxes such that every box has exactly three cubes and contains cubes of at most two different colors. If you have six yellow (Y) cubes, four green (G) cubes, one blue (B) cube, and one magenta (M) cube, here's one way to distribute them:



If you have four yellow (Y) cubes, four green (G) cubes, two blue (B) cubes, and two magenta (M) cubes, you could distribute them this way:



The result you're proving in this problem forms the basis for the alias method, a fast algorithm for simulating rolls of a loaded die. This has applications in machine learning (simulating different outcomes of a random event), operating systems (allocating CPU time to processes with different needs), and computational linguistics (generating random sentences based on differently-weighted rules).

* This problem adapted from Exercise 3.4.1.7 of *The Art of Computer Programming, Third Edition, Volume II: Seminumerical Algorithms* by Donald Knuth.